sufficient here to regulate the length of the template arm close to the curvilinear boundary. This possibility is ensured by simple change in the rules of random particle motion over orthogonal trajectories. In Fig. 1, this adaptive FDM template is "tied" to point B. If the adaptive template (AT) is also systematically rotated around the point B, the contributions of other boundary points may be taken into account. As in SRM, several stop frames are used to achieve acceptable accuracy, with subsequent averaging of the results. Table 1 gives the results of calculations using AT for four stop frames. Comparison of the two simplified approaches shows that the additional path in the random-motion scheme enriches the information at the given point and, with the same number of stop frames, increases the calculation accuracy, as a rule.

NOTATION

U(M), temperature at point M of region Ω; $\partial\Omega$, boundary of region; U_β, temperature at boundary point β; m_β/m, relative frequency of absorption of moving particle at point β; Φ_β, baricentric coordinates of point A in simplex.

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THERMAL HYSTERESIS IN NONLINEAR MEDIA

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One-dimensional thermal relaxation processes in media with nonlinear thermophysical properties are treated. Dynamic hysteresis is investigated theoretically in continuous and discontinuous nonstationary temperature fields. Boundary conditions are analyzed, for which a high-flow hysteresis process is realized. A quantitative estimate is given of irreversible variations in the material thermal state. Examples are given of constructing hysteresis branches. New properties are established for thermal shock waves propagating along the relaxing background.

It is well-known that hysteresis effects are observed in various physical processes (magnetic hysteresis [1], elastic hysteresis [2], etc.), and are characterized by a nonunique dependence between the quantities determining the material state and the external conditions of action. As applied to heat and mass transfer, these effects were noted in [3-6]. The mathematical methods of analyzing systems with nonlinear hysteresis were discussed in [7].

The purpose of the present study is construction of examples of analytic description of dynamic thermal hysteresis, realized during fast flow processes, both in continuous and discontinuous thermal fields. The mathematical model is based on the energy equation and on the generalized Fourier law [8, 9], written in dimensionless form

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$$cT_t + q_x + \frac{\nu}{x}\,q = 0, \quad q + \lambda T_x + \gamma q_t = 0, \tag{1}$$

where $q = \bar{q}/q_b$, $x = \bar{x}/x_b$, $\gamma = \bar{\gamma}/t_b$, etc. The variable scales are related by $\lambda_b T_b = q_b x_b$, $(x_b/t_b)^2 = \lambda_b/t_b c_b$, guaranteeing invariance of the dimensional and dimensionless forms of (1).

1. Hysteresis in a Continuous Thermal Wave. For $\nu = 0$, $\gamma \equiv$ const Eqs. (1) have an exact solution [10, 11], containing one arbitrary function $\zeta(\omega)$:

$$a = a_1(u + k_1)^{-2}, \quad a_1 = f_1^2/\gamma, \quad u'(T) = c, \quad a = \lambda/c; \quad a_1,\ f_1,\ k_1 - \text{const}, \tag{2}$$

$$x(u, \tau) = [f_1/(u + k_1)] + \zeta'(\omega), \quad \omega = (u + k_1)/\tau, \quad \tau = \exp(-t/\gamma), \tag{3}$$

$$\gamma q(u, \tau) = f_1 - \tau\zeta(\omega) + (u + k_1)\,\zeta'(\omega). \tag{4}$$

In what follows, to denote the arguments of the function ζ we use the letters ω, ε, z; in all these cases the symbol ζ is retained. In relation (2), for $k_1 = 0$ one can use for the thermophysical properties the power law dependences $\lambda = \lambda_1 T^{n_1}$, $c = c_1 T^{n_2}$, $n_1 + n_2 = -2$, $\lambda_1 c_1 \gamma = (1 + n_2)^2 f_1^2$. If $k_1 \neq 0$, we have from (2)

$$\lambda = \gamma c\,[f_2 + (\Lambda/f_1)]^2, \quad f_2 = -f_1/\gamma k_1, \quad \Lambda'(T) = \lambda(T).$$

For example:

$$\lambda = \lambda_1 \exp(n_1 T), \quad c = c_1 \exp(-n_1 T), \quad k_1 n_1 = -c_1, \quad \lambda_1 c_1 \gamma = (n_1 f_1)^2.$$

From the fixed wall

$$x_w = 0, \quad \gamma q_w = -\tau_w \zeta(\varepsilon), \quad \tau_w = -f_1 (d\zeta/d\varepsilon)^{-1}, \quad \varepsilon = \ln[(u_w + k_1)/\tau] \tag{5}$$

and along the relaxing background ($f_1 > 0$, $|x| < \infty$, $a < \infty$)

$$u_0 + k_1 = f_1(x + l_1)^{-1}, \quad \gamma q_0 = f_1 + \tau\,[\omega_0 \zeta'(\omega_0) - \zeta(\omega_0)], \quad l_1 = -\zeta'(\omega_0)$$

let there propagate a continuous thermal wave to the right

$$\omega = \omega_0 = \text{const}, \quad x_0 + l_1 = f_1/\omega_0 \tau, \quad x_0^0 = 0.$$

If at the boundary $x = H_1 > 0$ the temperature is given by the equality $u_0(H_1) + k_1 = \omega_0 \delta$, then the distance H_1 occurs in the wave after a time

$$t_1 = -\gamma \ln \delta, \quad H_1 \omega_0 = f_1(\delta^{-1} - 1), \quad 0 < \delta < 1. \tag{6}$$

It is implied that the solution (2)-(5) can be considered on a finite time interval $[0, t_1]$, equal to the duration of several relaxation periods γ. The selection of the functions $\zeta(\varepsilon)$ characterizes the parametric representation (5) of the thermal regime at the wall.

We analyze the relation between the temperature gradient and the specific flow along the isotherm $u(T_i) + k_1 = b > 0$, T_i = const for the case

$$\left(\frac{\partial x}{\partial u}\right)_i = B + A\cos(lz + \beta), \quad z = \ln \omega_i, \quad B < 0,\ A > 0,\ l > 0.$$

From Eqs. (3), (4) we find ($\psi_1 \equiv$ const, $B_1 - bB = f_1/b$):

$$\zeta(z) = [\psi_1 + (z - 1)\,B_1 + Ab(\sin y - l\cos y)\,l^{-1}(1 + l^2)^{-1}]\exp z, \tag{7}$$

$$\gamma q_i = B_1 b + f_1 + Ab^2(\cos y + l\sin y)(1 + l^2)^{-1},$$

$$y = lz + \beta = \alpha + (lt/\gamma), \quad (\alpha - \beta)/l = \ln b, \quad p = l\varepsilon + \beta. \tag{8}$$

Hence we obtain a simple relation between the isotherm rate $N_i > 0$ and the temperature gradient $(T_x)_i = (u_x)_i/c_i$:

$$\gamma N_i - B_1 = b\left[\left(\frac{\partial x}{\partial u}\right)_i - B\right] = Ab\cos y. \tag{9}$$

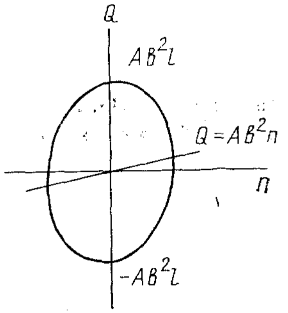

Fig. 1. Scheme of dynamic branch of thermal hysteresis at the isotherm.

This implies that the expression for N_i consists of a constant term and a function varying with time harmonically. Taking into account (6), we assume that the period ϑ of these oscillations does not exceed γ: $l > 2\pi$, $t_b < \bar{\gamma}$. Considering the expression

$$Q \equiv (\gamma q_i - f_1 - bB_1)(1 + l^2) = Ab^2 (n \pm l \sqrt{1 - n^2}), \quad n = \cos y,$$

we see that the branch of dynamic thermal hysteresis in the (n, Q) plane has the form of an ellipse (Fig. 1) with area $S_i = \pi A b^2 l$.

A quantitative estimate of the irreversible variations of the thermal state of the body during a period ϑ can also be obtained by using the expression for the entropy production per second per unit volume [12]: $\sigma = (q + \gamma q_t)(1/T)_x$. Choosing the relaxation component in this equation

$$\tilde{\sigma} = \gamma q_t (1/T)_x \tag{10}$$

along the isotherm, we find that the area of the hysteresis branch equals

$$S_i = \frac{A(1 + l^2)}{2} \left| \int_0^{\vartheta} \tilde{\sigma}_i dt \right| c_i T_i^2.$$

The boundary conditions at the wall, for which the dependences (8), (9) are realized, are described by Eqs. (5), (7) for $\varepsilon \in [\varepsilon^0, 0]$, $\varepsilon^0 = \ln b < 0$, while the solution possesses the properties:

$$q_w > 0, \quad u_w'(t) > 0, \ -f_1 \left(\psi_1 + \frac{Ab}{l}\right)^{-1} - k_1 > u_w \geqslant u_w^0 = u_0^0,$$

$$0 < N_i < w_0, \quad \partial x/\partial u < 0, \quad \psi_1 = -(f_1/b) - B_1\varepsilon^0 - Abl^{-1}\sin p^0 < 0.$$

The quantities A, B satisfy the system of inequalities

$$Ab < B_1 < [(f_1/b) - Ab(1 + 2l^{-1})](1 - \varepsilon^0).$$

From (5), (7), we obtain the equation

$$\frac{\gamma q_w}{u_w + k_1} = B_1(1 - \varepsilon) - \psi_1 + \frac{Ab}{l(1 + l^2)}(l\cos p - \sin p),$$

in which the last term is a nonmonotonic function, determining the mutual relation between the parameters Ab, l of the harmonic component of the isotherm rate (9).

2. Thermal Field Properties at a Strong Discontinuity Front. The conditions of dynamic compatibility [13] at discontinuity lines $x = x_j(t)$ of the thermal field are [14, 15]

$$V_j - V_* = N_j(q_j - q_*), \quad q_j - q_* = N_j(u_j - u_*), \quad N_j = x_j'(t), \tag{11}$$

where $V'(T) = \lambda/\gamma$. We use for u(x, t), q(x, t) equations of the type

$$\frac{du(x_j, t)}{dt} = (u_t + Nu_x)_j$$

and, acting similarly to [16], we obtain from the heat transport equation (1) and conditions (11) expressions for the first derivatives with respect to coordinate

$$(u_x)_j = (N_j^2 - w_j^2)^{-1} \left[\frac{q}{\gamma} + \frac{dq}{dt} + N \left(\frac{du}{dt} + \frac{\nu q}{x} \right) \right]_j, \quad (12)$$

$$(q_x)_j = (N_j^2 - w_j^2)^{-1} \left[N \left(\frac{q}{\gamma} + \frac{dq}{dt} \right) + w^2 \left(\frac{du}{dt} + \frac{\nu q}{x} \right) \right]_j, \quad N_j^2 \neq w_j^2. \quad (13)$$

The relaxing thermal field ahead of the discontinuity is selected in the form $u_* \equiv$ const, $q_* = b_* x^{-\nu} \exp(-t/\gamma_*)$, $\gamma_* = \gamma(T_*)$. Besides, for the one-dimensional planar case ($\nu = 0$) on the background can be the inhomogeneous temperature field $a(u_*)(du_*/dx) = -b_*$, $q_* = b_* + b_1 \exp(-t/\gamma)$, $|x| < \infty$, b_*, b_1, γ – const.

On the (u, q) plane the analog of the shock adiabat [13] is the curve $H(u_j, q_j) = 0$, being the set of u_j, q_j values satisfying conditions (11) at the discontinuity for $q_* = 0$, $u_* =$ const and some arbitrary N_j. Let s be the path length of this line. Using the result of [17], it can be verified that on the curve H = 0 the relations $dN_j/ds = 0$, $N_j = w_j$, being the analogs of Jouguet conditions of detonation theory, are satisfied only one at a time. If $N_j - w_j$ on the curve H = 0 changes sign at the point (u_j, q_j), at this point N_j has an extremum. The opposite statement is also correct [17].

Consider nonlinear media, whose thermophysical properties on both sides of the discontinuity are described by the relation

$$w^2 \equiv \lambda/c\gamma = w_1 + 2w_2 u, \quad w_2 \neq 0, \quad T \in [T_1, T_2]. \quad (14)$$

This variant covers two important special cases: 1) $w_1 = 0$, i.e., the parameters $\lambda \sim u^{d_1}$, $c \sim u^{d_2}$, $\gamma \sim u^{d_3}$ are uniform power-law functions of temperature, $d_1 = 1 + d_2 + d_3$, where, for example, the values $d_1 = 5/2$, $d_2 = 0$, $d_3 = 3/2$ correspond to a completely ionized plasma, traveling waves, and discontinuities in these media, as investigated in [18, 19]; 2) linear temperature dependence of the heat conduction coefficient

$$\lambda = \lambda_1 + \lambda_2 T; \quad c, \ \gamma - \text{const}. \quad (15)$$

This approximation can be applied to many materials. It follows from (11), (14) that the right-hand sides in (12), (13) are evaluated by means of the equations

$$u_j + u_* = (N_j^2 - w_1)/w_2, \quad (q_j - q_*)/N_j = [(N_j^2 - w_1)/w_2] - 2u_*. \quad (16)$$

Hence, in particular, we have

$$N_j^2 - w_j^2 = w_2 (u_* - u_j) = w_*^2 - N_j^2, \quad N_j^2 = (w_*^2 + w_j^2)/2.$$

This implies that in the class of media (14) the squared velocity of the thermal shock wave is the arithmetic mean of the squared velocities of thermal perturbations ahead and following the discontinuity front. The Jouguet point is absent in using relation (14).

If $w_1 = 0$ and ahead of the discontinuity front the background is "cold," $u_* = 0$, $q_* = 0$, then

$$w_2 (u_x)_j = -\left[\frac{N}{\gamma} + 5N'(t) + \nu \frac{N^2}{x} \right]_j, \quad w_2 (q_x)_j =$$

$$= -N_j \left[\frac{N}{\gamma} + 7N'(t) + 2 \frac{\nu N^2}{x} \right]_j.$$

Hence follow the following qualitative conclusions: The geometric parameter ν appears in terms depending on jump rates obeying a power law; if $N_j(t) \leq N_0 < \infty$, $t \geq 0$, upon removal of the discontinuity front from the center of cylindrical or spherical symmetry the effect of ν is monotonically reduced.

For $\nu = 0$, c, γ – const the generalization of relations (12), (13) to the case of derivatives of order $n \geq 1$ is

$$r_{n+1} = (N_j^2 - w_j^2)^{-1} \left[N_j \frac{dr_n}{dt} + \frac{ds_n}{dt} + (s_n + 2\gamma w_2 R_n)\gamma^{-1} \right], \quad r_n = \left(\frac{\partial^n u}{\partial x^n} \right)_j, \quad (17)$$

$$s_{n+1} = (N_j^2 - w_j^2)^{-1}\left[w_j^2 \frac{dr_n}{dt} + N_j \frac{ds_n}{dt} + N_j (s_n + 2\gamma w_2 R_n)\gamma^{-1}\right],$$
$$s_n = \left(\frac{\partial^n q}{\partial x^n}\right)_j, \quad 2a_2 R_n = r_1 r_n + \frac{\partial}{\partial x}\sum_{m=0}^{n-2} \frac{\partial^m}{\partial x^m}(r_1 r_{n-m-1}). \tag{18}$$

Thus, if the displacement velocity of the thermal shock wave is determined experimentally, and one knows the parameters of the background over which this wave propagates, relations (12), (13), (17), (18) make it possible to calculate the temperature and thermal flux behind the discontinuity front.

3. Thermal Hysteresis at Strong Discontinuity Front. Let a one-dimensional planar ($\nu = 0$) strong discontinuity of the thermal field be displaced with velocity

$$N_j = B + A\cos(kt + \beta), \quad B > A > 0, \; k > 0 \tag{19}$$

in a nonlinear medium with thermophysical properties (14), (15); the background parameters are: $u_* = \text{const}$, $q_* = 0$; initially the discontinuity is located at the wall

$$x_w = 0, \quad q = q_w(t), \quad t \geqslant 0. \tag{20}$$

It is seen from (16) that for $w_2 > 0$ we have a heating shock wave, $u_j > u_*$, while for $w_2 < 0$ we have a cooling wave, $u_j < u_*$ [14].

Using Eqs. (12), (13), (17), (18), we find the temperature field behind the jump front:

$$q - q_j = \sum_{n=1}^{\infty} \frac{s_n}{n!}(x - x_j)^n, \quad u - u_j = \sum_{n=1}^{\infty} \frac{r_n}{n!}(x - x_j)^n. \tag{21}$$

These series are analytic functions for all $|x - x_j| \geq 0$, $t \geq 0$, if the following conditions are satisfied

$$\varkappa = \left(\frac{10}{\Delta}\right)^2 k\mu^3 < 1, \quad \frac{1}{\gamma} < k < 1, \quad \mu = B + A, \quad \Delta = (B - A)^2 - w_*^2 > 0. \tag{22}$$

Choosing the first M terms in (21), we obtain that the residuals of these series are bounded from above by the absolute value of the expression

$$D\frac{k\mu^4}{|w_2|\,\Delta}\left|\exp(x_j - x) - 1 - \sum_{n=1}^{M}\frac{(x_j - x)^n}{n!}\right|,$$

in which D equals, respectively, 8 and 6. For values $|x - x_j| \leq 1$ we find that the absolute error of the discarded terms does not exceed $[Dk\mu^4\varkappa^M/|w_2|\Delta(1 - \varkappa)]$, $M \geq 1$. The restrictions (22) in the set with discontinuity stability conditions [13] $w_*^2 < N_j^2 < w_j^2$ provide the following estimates of the parameters A, B, k, w_*, for which this thermal process is realized ($t_b < \bar{\gamma}$):

1) if $\Delta > 10$, $k\mu^3 < 1$, then $\gamma^{-1} < k < \mu^3$,

$$0 < w_*^2 < \min\{(B - A)^2 - 4AB;\; (B - A)^2 - 10\}; \tag{23}$$

2) if $\Delta > 10$, $k\mu^3 > 1$, then

$$\max\{\gamma^{-1}, \mu^{-3}\} < k < \left(\frac{\Delta}{10}\right)^2 \mu^{-3},$$

and inequality (23) remains in effect;

3) if $\Delta < 10$, $\varkappa < 1$, then

$$\gamma^{-1} < k < \left(\frac{\Delta}{10}\right)^2 \mu^{-3}, \quad 0 < (B - A)^2 - 10 < w_*^2 < (B - A)^2 - 4AB.$$

Consider the variables $\psi = w_2(u_x)_j$, $\varphi = w_2 q_j$, characterizing the temperature and thermal flux gradients along the strong discontinuity lines. In the ψ, φ plane the dynamic hysteresis branch is represented parametrically:

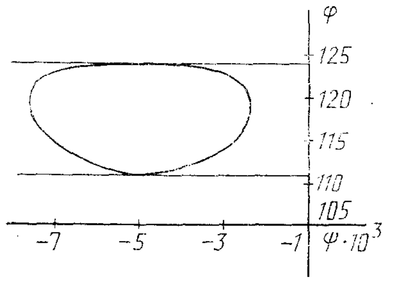

Fig. 2. Example of dynamic thermal hysteresis branch at strong discontinuity front (dimensionless quantities).

$$N_j \in [B-A,\ B+A],$$

$$\psi = -\frac{N_j}{\gamma} \pm \frac{k(5N_j^2 - w_*^2)}{(N_j^2 - w_*^2)} \sqrt{A^2-(N_j-B)^2}\,, \quad \varphi = N_j(N_j^2 - w_*^2).$$

Its shape for $A = 0.1$, $B = 5$, $w_* = 1$, $k = 5 \cdot 10^{-3}$, $\gamma = 10^3$ is shown in Fig. 2. The branch area is described quite accurately by the equation

$$S_j \simeq \pi A^2 kh \left[\frac{5}{4}(4B^2 + A^2) - w_*^2\right].$$

To simplify the calculation it was taken into account that the function $h = (3N_j^2 - w_*^2)/(N_j^2 - w_*^2)$ varies little over the interval investigated. In particular, for this example the deviation of $h(N_j)$ from the approximate constant value $h \approx 3.083388$ does not exceed 0.115%. Applying (10) and integrating over a time segment equal to the oscillation period of the discontinuity rate, we obtain

$$S_j \simeq \frac{w_2^2}{\gamma c}\left|\int\limits_0^{\vartheta} (\tilde{\sigma}u^2)_j\, dt\right|.$$

The thermal flux at the wall (20) is obtained from (21) for $x = 0$, with $q_w^0 = q_j^0$. It is important that in the first approximation already the expression for $q_w(t)$ contains the resonance term $2B^2Ahkt\sin(kt + \beta)$, characterizing the external thermal action on the medium.

Conclusions. For continuous and discontinuous temperature fields one can show the following features common for all of them, for which dynamic thermal hysteresis is realized: 1) fast flow processes with time scales shorter than the relaxation period of the thermal flux; 2) the velocity of the moving line (isotherm, jump front), on which irreversible variations occur of the thermal state of the material, has a high-frequency harmonic component; 3) the area of the hysteresis branch is uniquely related to the relaxation component of the entropy output (10).

The results of the present study are useful for theoretical analysis of thermal processes in a number of contemporary energy devices, operating under substantially nonstationary conditions, when the possibilities of the classical model of thermal conductivity are too restricted for explaining fast-flowing nonlinear effects.

NOTATION

x, spatial coordinate; t, time; T, temperature; q, specific thermal flux; λ, thermal conductivity coefficient; c, specific bulk heat capacity; γ, relaxation time of the thermal flux; a, temperature conductivity; w, propagation rate of small thermal perturbations; N, displacement velocity of the moving line; S, area of the hysteresis branch; σ, entropy product per second per unit volume; $\nu = 0, 1, 2$, a geometric parameter, corresponding to planar, cylindrical, and spherical symmetry; and ϑ, oscillation period. The indices are: a bar over a symbol is for dimensional quantities; i, j, functional values on the isotherm and on the strong discontinuity line; b, scale of dimensional quantities; *, for medium parameters ahead of the discontinuity front; w, fixed wall; ^{0}, initial value at $t = 0$; $_0$, for parameters on the thermal wave; the prime is ordinary differentiation with respect to the argument in parenthesis; and the independent variable in the subscript stands for partial differentiation.

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